# Convergence of the piecewise orthogonal collocation for periodic solutions of retarded functional differential equations

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# Overview

# Coupled RE/DDE equations

Coupled systems of Renewal Equations and Delay Differential Equations are of the form

$$\begin{cases} x(t) = F(x_t, y_t), \\ y'(t) = G(x_t, y_t) \end{cases}$$

where, for a given *delay* $\tau > 0$ and *state spaces* $\mathtt{X}$ and $\mathtt{Y}$,

$$x_t \in \mathtt{X}, \quad y_t \in \mathtt{Y}, \quad x_t(\theta) = x(t+\theta) \text{ for all } \theta \in [-\tau, 0],$$

and $y_t$ is defined similarly. $F$ and $G$ are autonomous, smooth, typically non-linear functions such that $F$ is integral in $x$. The state spaces are Banach spaces of $\mathbb{R}$-valued functions defined in $[-\tau, 0]$.

Goal: compute periodic solutions.

# Periodic Boundary Value Problems

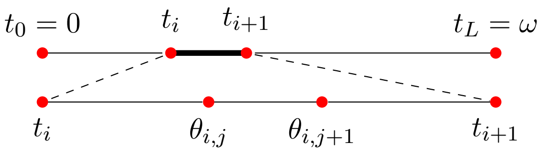
A periodic solution of a coupled system with period $\omega > 0$ can be obtained by solving a BVP of the form

$$\begin{cases} x(t) = F(x_t, y_t), & t \in [0, \omega], \\ y'(t) = G(x_t, y_t), & t \in [0, \omega], \\ (x_0, y_0) = (x_\omega, y_\omega), & \text{(periodicity condition)} \\ p(x|_{[0,\omega]}, y|_{[0,\omega]}) = 0 & \text{(phase condition)} \end{cases}$$

where $x$ and $y$ are defined in $[-\tau, \omega]$ and $p$ is a (usually linear) real-valued function, introduced in order to remove translational invariance [1].

[1] DOEDEL E., *Lecture notes on numerical analysis of nonlinear equations*, Numerical Continuation Methods for Dynamical Systems, Understanding Complex Systems, Dordrecht, 1–49, 2007.

# Piecewise polynomial collocation (extending [2])




For a given mesh $0 = t_0 < \cdots < t_L = \omega$, and collocation points $t_i < \theta_{i,1} \cdots < \theta_{i,m} < t_{i+1}$, look for $m$-degree continuous piecewise polynomials $u, v$ in $[0, \omega]$ such that

$$\begin{cases} u(\theta_{i,j}) = F(u_{\theta_{i,j}}, v_{\theta_{i,j}}), & 1 \leq j \leq m,\ 0 \leq i < L, \\ v'(\theta_{i,j}) = \omega G(u_{\theta_{i,j}}, v_{\theta_{i,j}}), & 1 \leq j \leq m,\ 0 \leq i < L, \\ (u(\theta_{i,j} - \omega), v(\theta_{i,j} - \omega)) = (u(\theta_{i,j}), v(\theta_{i,j})), & 1 \leq j \leq m,\ 0 \leq i < L, \\ p(u, v) = 0. \end{cases}$$

[2] ENGELBORGHS K., LUZYANINA T., IN 'T HOUT K.J., AND ROOSE D., *Collocation methods for the computation of periodic solutions of delay differential equations*, SIAM J. Sci. Comput., 22 (2001), pp. 1593–1609.

# Orthogonal collocation

Collocation points $t_i < \theta_{i,1} < \cdots < \theta_{i,m} < t_{i+1}$ are chosen as the roots of the $m$-th element of some family of orthogonal polynomials (e.g., Chebyshev, Gauss-Legendre) defined in $[t_i, t_{i+1}]$.

The discretization level is determined by both $m$ and the number $L$ of mesh intervals.

- Spectral Element Method (SEM), fixed $L$ and $m \to \infty$
- Finite Element Method (FEM), fixed $m$ and $L \to \infty$, typical in practical implementations [3, 4].

In the following, FEM is equivalent to $h := \frac{\omega}{L} \to 0$ (uniform mesh).

[3] MatCont, https://sourceforge.net/projects/matcont/.
[4] DDE-BIFTOOL, http://ddebiftool.sourceforge.net/.

# Phase conditions

A *trivial* phase condition is one of the form

$$x(0) = \bar{x} \quad \text{or} \quad y(0) = \bar{y},$$

where $\bar{x}, \bar{y}$ are fixed. An *integral* phase condition [1] is one of the form

$$\int_0^\omega \langle x(t), \tilde{x}'(t) \rangle \, \mathrm{d}t = 0 \quad \text{or} \quad \int_0^\omega \langle y(t), \tilde{y}'(t) \rangle \, \mathrm{d}t = 0,$$

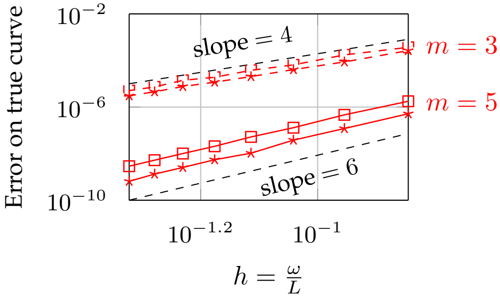
where $\tilde{x}$ and $\tilde{y}$ are given reference solutions.

[1] DOEDEL E., *Lecture notes on numerical analysis of nonlinear equations*, Numerical Continuation Methods for Dynamical Systems, Understanding Complex Systems, Dordrecht, 1–49, 2007.

# Does the FEM converge?

$$\begin{cases} x(t) = \dfrac{\gamma}{2}\displaystyle\int_{-3}^{-1} x(t+\theta)(1-x(t+\theta))\,\mathrm{d}\theta, \\ y'(t) = \gamma x(t)(x(t-1)(1-x(t-1)) - x(t-3)(1-x(t-3)) + y(t), \end{cases}$$

Uniform error for $m = 3, 5$ of the $x$-component (stars) and the $y$-component (squares). Gauss-Legendre collocation: $\mathcal{O}(h^{m+1})$.




Lack of a general, theoretical proof of convergence.

# Section 2

## Theoretical convergence for DDEs

[5] ANDÒ A., AND BREDA D., *Convergence analysis of collocation methods for computing periodic solutions of retarded functional differential equations*, SIAM J. Numer. Anal., to appear, https://arxiv.org/abs/2008.07604

# Time scaling for periodic BVPs

To compute a periodic solution of $y'(t) = G(y_t)$ of unknown period $\omega$, one can transform it into $y'(t) = \omega G(y_t \circ s_\omega)$, where $s_\omega(t) := \frac{t}{\omega}$. Two possible (equivalent) BVPs:

$$\begin{cases} y'(t) = \omega G(y_t \circ s_\omega), & t \in [0,1], \\ y_0 = y_1 \\ p(y|_{[0,1]}) = 0 \end{cases}$$

for $y$ defined in $[-1,1]$

$$\begin{cases} y'(t) = \omega G(\overline{y_t} \circ s_\omega), & t \in [0,1], \\ y(0) = y(1) \\ p(y) = 0 \end{cases}$$

for $y$ defined in $[0,1]$.

$\overline{y_t}$ is defined as:

$$\overline{y_t}(\theta) := \begin{cases} y(t+\theta), & t+\theta \in [0,1], \\ y(t+\theta+1), & t+\theta \in [-1,0). \end{cases}$$

# The Problem in Abstract Form

Maset [6] suggests a general approach to solve BVPs for neutral functional differential equations numerically, by restating such problems in the following abstract form:

**Problem in Abstract Form**

Let $\mathbb{V}$ be a normed space and $\mathbb{U}, \mathbb{A}, \mathbb{B}$ be Banach spaces. Let

$$\mathcal{F} : \mathbb{V} \times \mathbb{U} \times \mathbb{B} \to \mathbb{U}, \quad \mathcal{B} : \mathbb{V} \times \mathbb{U} \times \mathbb{B} \to \mathbb{A} \times \mathbb{B}, \quad \mathcal{G} : \mathbb{U} \times \mathbb{A} \to \mathbb{V}$$

be operators such that $\mathcal{G}$ is linear.
Find $(v, \beta) \in \mathbb{V} \times \mathbb{B}$ such that $v = \mathcal{G}(u, \alpha)$ for some $(u, \alpha) \in \mathbb{U} \times \mathbb{A}$ and

$$\begin{cases} u = \mathcal{F}(v, u, \beta), \\ \mathcal{B}(v, u, \beta) = 0. \end{cases}$$

[6] MASET S., *An abstract framework in the numerical solution of boundary value problems for neutral functional differential equations*, Numer. Math., 133 (2016) pp. 525–555

# An equivalent fixed point problem

**Problem in Abstract Form**

Find $(v, \beta) \in \mathbb{V} \times \mathbb{B}$ such that $v = \mathcal{G}(u, \alpha)$ for some $(u, \alpha) \in \mathbb{U} \times \mathbb{A}$ and

$$\begin{cases} u = \mathcal{F}(v, u, \beta), \\ \mathcal{B}(v, u, \beta) = 0. \end{cases}$$

In other words: find $x^* \in X := \mathbb{U} \times \mathbb{A} \times \mathbb{B}$ such that $x^* = \Phi(x^*)$ for $\Phi : X \to X$ given by

$$\Phi(x) := \begin{pmatrix} \mathcal{F}(\mathcal{G}(u, \alpha), u, \beta) \\ (\alpha, \beta) - \mathcal{B}(\mathcal{G}(u, \alpha), u, \beta) \end{pmatrix}$$

with $x := (u, \alpha, \beta)$.

# Search of periodic solutions as a **PAF**

Consider the space $B^\infty$ of measurable and bounded functions with uniform norm, and the space $B^{1,\infty}$ of functions having derivative in $B^\infty$, with norm $f \to \|f\|_\infty + \|f'\|_\infty$. Let $\mathbb{U} = B^\infty([0,1], \mathbb{R}^d)$, $\mathbb{A} = B^{1,\infty}([-1,0], \mathbb{R}^d)$, $\mathbb{V} = B^{1,\infty}([-1,1], \mathbb{R}^d)$, $\mathbb{B} = \mathbb{R}$. The **PAF** reads:

Find $(v^*, \omega^*) \in \mathbb{V} \times \mathbb{R}$ with $v^* = \mathcal{G}(u^*, \psi^*)$ for $\mathcal{G} : \mathbb{U} \times \mathbb{A} \to \mathbb{V}$ given by

$$\mathcal{G}(u,\psi)(t) = \begin{cases} \psi(0) \quad + \int_0^t u(s)\,\mathrm{d}s, & t \in [0,1], \\ \psi(t), & t \in [-1,0]. \end{cases}$$

and $(u^*, \psi^*, \omega^*)$ a fixed point of $\Phi : \mathbb{U} \times \mathbb{A} \times \mathbb{R} \to \mathbb{U} \times \mathbb{A} \times \mathbb{R}$ given by

$$\Phi(u,\psi,\omega) = \begin{pmatrix} \omega G(\mathcal{G}(u,\psi)_{(\cdot)} \circ s_\omega) \\ \mathcal{G}(u,\psi)_1 \\ \omega - p(\mathcal{G}(u,\psi)|_{[0,1]}) \end{pmatrix}.$$

# Theoretical assumptions needed by [6]

Need to prove the validity of the following:

- $\mathrm{A}\mathfrak{F}\mathfrak{B}$ $\Phi$ is Fréchet-differentiable
- $\mathrm{A}\mathfrak{G}$ $\mathcal{G}$ is bounded
- $\mathrm{A}x^*1$ $D\Phi$ is locally Lipschitz continuous at $x^* = (u^*, \psi^*, \omega^*)$ (fixed point, lying in a more regular subspace)
- $\mathrm{A}x^*2$ the operator $I - D\Phi(x^*)$ is invertible with bounded inverse.

General fixed point operator theory in [7].

[6] MASET S., *An abstract framework in the numerical solution of boundary value problems for neutral functional differential equations*, Numer. Math., 133 (2016) pp. 525–555

[7] KRASNOSEL'SKII M. A., VAINIKKO G. M., ZABREYKO R. P., R. Y. B, AND STET'SENKO V. V., *Approximate Solution of Operator Equations*, Springer Netherlands, 1 edition (1972)

# Regularity hypotheses

1. $\mathtt{Y} = B^{\infty}([-\tau, 0], \mathbb{R}^d)$ and $Y = B^{\infty}([-1, 0], \mathbb{R}^d)$
2. $\mathbb{U} = B^{\infty}([0, 1], \mathbb{R}^d)$, $\mathbb{V} = B^{1,\infty}([-1, 1], \mathbb{R}^d)$, $\mathbb{A} = B^{1,\infty}([-1, 0], \mathbb{R}^d)$
3. $G : \mathtt{Y} \to \mathbb{R}^d$ is Fréchet-differentiable at every $\mathtt{y} \in \mathtt{Y}$
4. $G \in \mathcal{C}^1(\mathtt{Y}, \mathbb{R}^d)$
5. there exist $r > 0$ and $\kappa \geq 0$ such that

$$\|DG(\mathtt{y}) - DG(v_t^* \circ s_{\omega^*})\|_{\mathbb{R}^d \leftarrow \mathtt{Y}} \leq \kappa \|\mathtt{y} - v_t^* \circ s_{\omega^*}\|_{\mathtt{Y}}$$

for every $\mathtt{y} \in \overline{B}(v_t^* \circ s_{\omega^*}, r)$, uniformly with respect to $t \in [0, 1]$
$\Rightarrow$ no state-dependent delays

$1, 2, 3 \Rightarrow \mathrm{A}\mathfrak{F}\mathfrak{B}$, $\quad 2 \Rightarrow \mathrm{A}\mathfrak{G}$, $\quad 1, 2, 3, 5 \Rightarrow \mathrm{A}x^*1$,
$1, 2, 4$, *hyperbolicity* of the concerned periodic solution $\Rightarrow \mathrm{A}x^*2$

# Notation to describe the linearized problem

The Fréchet differential of the right-hand side of the equation can be written as $D\mathcal{F}(\hat{v},\hat{u},\hat{\omega})(v,u,\omega) = \mathfrak{L}(\cdot;\hat{v},\hat{\omega})[v_{\cdot} \circ s_{\hat{\omega}}] + \omega\mathfrak{M}(\cdot;\hat{v},\hat{\omega})$, where

$$\mathfrak{L}(t;v,\omega) := \omega DG(v_t \circ s_\omega)$$

and

$$\mathfrak{M}(t;v,\omega) := G(v_t \circ s_\omega) - \mathfrak{L}(t;v,\omega)[v_t' \circ s_\omega] \cdot \frac{s_\omega}{\omega}.$$

Let $\mathfrak{L}^* := \mathfrak{L}(\cdot;v^*,\omega^*)$, $\mathfrak{M}^* := \mathfrak{M}(\cdot;v^*,\omega^*)$, and $T^*(1,0) : Y \to Y$ the *monodromy operator* of the linear homogeneous DDE

$$y'(t) = \mathfrak{L}^*(t)[y_t \circ s_{\omega^*}].$$

# Proof of A$x^*$2 in *generic* case

**Proposition**

For all $(u_0, \psi_0, \omega_0) \in \mathbb{U} \times \mathbb{A} \times \mathbb{B}$ there exists a unique $(u, \psi, \omega) \in \mathbb{U} \times \mathbb{A} \times \mathbb{B}$ such that

$$\begin{cases} u = \mathfrak{L}^*[\mathcal{G}(u,\psi)_\cdot \circ s_{\omega^*}] + \omega\mathfrak{M}^* + u_0 \\ \psi = \mathcal{G}(u,\psi)_1 + \psi_0 \\ p(\mathcal{G}(u,\psi)|_{[0,1]}) = \omega_0. \end{cases} \tag{1}$$

The first of (1) can be viewed as the IVP for $v = \mathcal{G}(u, \psi)$

$$\begin{cases} v'(t) = \mathfrak{L}^*(t)[v_t \circ s_{\omega^*}] + \omega\mathfrak{M}^*(t) + u_0(t) \\ v_0 = \psi. \end{cases} \tag{2}$$

# Proof of A$x^*$2 in *generic* case

VCF: $v_t = T^*(t,0)\psi + \int_0^t [T^*(t,s)X_0][\omega\mathfrak{M}^*(s) + u_0(s)]\,\mathrm{d}s$ [8]
BC in (1): $\psi = T^*(1,0)\psi + \int_0^1 [T^*(1,s)X_0][\omega\mathfrak{M}^*(s) + u_0(s)]\,\mathrm{d}s + \psi_0$.

Hyperbolicity: $1$ is a simple eigenvalue of $T^*(1,0) \Rightarrow$ decomposition of the state space as $R \oplus K$, with $R$ the range of $I - T^*(1,0)$ and $K$=span$\{\varphi\}$ its kernel. Let

$$\xi_1^* := \int_0^1 [T^*(1,s)X_0]\mathfrak{M}^*(s)\,\mathrm{d}s = r_1 + k_1\varphi,$$
$$\xi_2^* := \int_0^1 [T^*(1,s)X_0]u_0(s)\,\mathrm{d}s + \psi_0 = r_2 + k_2\varphi.$$

Then $\omega\xi_1^* + \xi_2^* = \psi - T^*(1,0)\psi \in R \Rightarrow \omega = -k_2/k_1$ as long as $k_1 \neq 0$.

[8] Hale J.K., *Theory of functional differential equations*, Number 99 in Applied Mathematical Sciences. Springer Verlag, New York, first edition (1977).

# $k_1 \neq 0$: Riemann-Stieltjes integrals

$$\xi_1^* := \int_0^1 [T^*(1,s)X_0]\mathfrak{M}^*(s)\,\mathrm{d}s = r_1 + k_1\varphi.$$

Assume $k_1 = 0$. Then $\xi_1^* \in R \Rightarrow y'(t) = \mathfrak{L}^*(t)[y_t \circ s_{\omega^*}] + \mathfrak{M}^*(t)$ has a 1-periodic solution.

Riemann-Stieltjes: $\mathfrak{L}^*(t)[\psi \circ s_{\omega^*}] = \int_{-\tau}^0 \mathrm{d}_\sigma \mathtt{n}^*(t,\sigma)\psi(s_{\omega^*}(\sigma))$.

$$\begin{aligned}\mathfrak{M}^*(t) &= G(v_t^* \circ s_{\omega^*}) - \mathfrak{L}^*(t)[v^{*\prime}_t \circ s_{\omega *}] \cdot \tfrac{s_{\omega^*}}{\omega^*} \\ &= G(v_t^* \circ s_{\omega^*}) - \int_{-\tau}^0 \mathrm{d}_\sigma \mathtt{n}^*(t,\sigma) v^{*\prime}(t + s_{\omega^*}(\sigma)) \cdot \frac{s_{\omega^*}(\sigma)}{\omega^*} \\ &= \frac{1}{\omega^*}\left(v^{*\prime}(t) - \int_{-r}^0 \mathrm{d}_\theta n^*(t,\theta) v^{*\prime}(t+\theta)\theta\right).\end{aligned}$$

# $k_1 \neq 0$: adjoint theory [8] (Fredholm theory [9])

$y'(t) = \int_{-r}^{0} \mathrm{d}_\theta n^*(t,\theta) y(t+\theta) = \mathfrak{L}^*(t)[y_t \circ s_{\omega^*}]$ has exactly one 1-periodic solution (viz. $v^{*\prime}$). Then its adjoint equation

$$z(t) + \int_t^\infty z(\theta) n^*(\theta, t-\theta)\,\mathrm{d}\theta = \text{constant},$$

also has exactly one 1-periodic solution, say $z^*$. Bilinear form:

$$(z^t, y_t)_t := z(t)y(t) + \int_{-r}^{0} \mathrm{d}_\beta \left[ \int_0^r z(t+\xi) n^*(t+\xi, \beta-\xi)\,\mathrm{d}\xi \right] y(t+\beta).$$

By [8, pag. 200] $({z^*}^t, {v_t^*}')_t = c^* \neq 0$ (left and right eigenvectors of simple eigenvalues are not orthogonal to each other).

By [8, Sect. 9.1] $\int_0^1 z^*(t)\mathfrak{M}^*(t)\,\mathrm{d}t = 0$.

[8] Hale J.K., *Theory of functional differential equations*, Number 99 in Applied Mathematical Sciences. Springer Verlag, New York, first edition (1977).

[9] Kress R., *Linear integral equations*, Number 82 in Applied Mathematical Sciences. Springer-Verlag, New York (1989).

# $k_1 \neq 0$: conclusion by contradiction

$$
\begin{aligned}
c^* = \int_0^1 c^* dt = & \int_0^1 z^*(t) v^{*\prime}(t) dt \\
& + \int_0^1 \int_{-r}^0 \mathrm{d}_\beta \left[ \int_0^r z^*(t+\xi) n^*(t+\xi, \beta - \xi)\, \mathrm{d}\xi \right] v^{*\prime}(t+\beta)\, \mathrm{d}t \\
= & \int_0^1 z^*(t) v^{*\prime}(t) dt - \int_0^1 z^*(t) \int_{-r}^0 \mathrm{d}_\theta n^*(t,\theta) v^{*\prime}(t+\theta)\theta\, \mathrm{d}t \\
= & \int_0^1 z^*(t) \left( v^{*\prime}(t) - \int_{-r}^0 \mathrm{d}_\theta n^*(t,\theta) v^{*\prime}(t+\theta)\theta \right) \mathrm{d}t \\
= & \, \omega^* \int_0^1 z^*(t) \mathfrak{M}^*(t)\, \mathrm{d}t = 0
\end{aligned}
$$

# Conclusion of proof of A$x^*$2

Let $\eta$ be such that $\omega\xi_1^* + \xi_2^* = \eta - T^*(1,0)\eta$.
Every $\psi$ satisfying $\omega\xi_1^* + \xi_2^* = \psi - T^*(1,0)\psi$ is $\eta + \lambda\varphi$ for some $\lambda \in \mathbb{R}$.
The value of $\lambda$ is fixed by imposing the second boundary condition in (1), i.e., $p(v(\cdot;\eta)|_{[0,1]}) + \lambda p(v(\cdot;\varphi)|_{[0,1]}) = \omega_0$.
Uniqueness follows from $p(v(\cdot;\varphi)|_{[0,1]}) \neq 0$.

□

Note that the operator $I - D\Phi(x^*)$ is not invertible with the classic BVP formulation: given $\alpha_0 \in \mathbb{R}$, finding $\alpha \in \mathbb{R}$ such that

$$\alpha = \mathcal{G}(u,\alpha) + \alpha_0$$

is not possible if $\int_0^1 u(s)\,\mathrm{d}s = 0$.

# Role of $\omega$ and consequent challenges

$\mathrm{A}\mathfrak{F}\mathfrak{B}$ $\Phi$ is Fréchet-differentiable

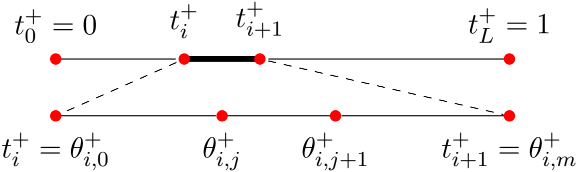
- $D_\omega G(\mathcal{G}(u,\psi)_{(\cdot)} \circ s_\omega)$ requires $\mathbb{V}$ to be at least as regular as $B^{1,\infty}$
- The derivatives of the elements of $\mathbb{V}$ cannot all be continuous at $0$, so $\mathbb{V}$ cannot be as regular as $C^1$
- Time derivatives are intended from the right (typical with DDEs)

$\mathrm{A}x^*1$ In the classic BVP formulation, $\mathcal{G}(u,\alpha) := \alpha + \int_0^\cdot u(s)\,\mathrm{d}s$ and $\overline{\mathcal{G}(u,\alpha)_{(\cdot)}}$ is not even continuous, unless $\mathbb{U}$ only consists of functions having zero mean. In [10] the problem is dealt with by considering

$$\mathcal{G}(u,\alpha) := \alpha + \int_0^t [Q_0 u](s)\,\mathrm{d}s, \qquad [Q_0 u](t) = u(t) - \int_0^1 u(s)\,\mathrm{d}s.$$

[10] SIEBER J., *Finding periodic orbits in state-dependent delay differential equations as roots of algebraic equations*, Discrete Contin. Dyn. S. Ser. S, 32(8):2607–2561 (2012).

# Discretization




Restriction and prolongation operators $\rho_L^+ : \mathbb{U} \to \mathbb{U}_L$, $\pi_L^+ : \mathbb{U}_L \to \mathbb{U}$ in $[0,1]$: for $u_L \in \mathbb{U}_L$, $\pi_L^+ u_L \in \mathbb{U}$ is the unique element of the relevant space of continuous piecewise polynomials such that

$$\pi_L^+ u_L(0) = u_{1,0}, \quad \pi_L^+ u_L(\theta_{i,j}^+) = u_{i,j}, \quad j = 1, \ldots, m, \; i = 1, \ldots, L.$$

$\rho_L^- : \mathbb{A} \to \mathbb{A}_L$ and $\pi_L^- : \mathbb{A}_L \to \mathbb{A}$ are defined similarly in $[-1, 0]$, from

$$\theta_{i,j}^- = \theta_{i,j}^+ - 1, \quad j = 1, \ldots, m, \; i = 1, \ldots, L.$$

FEM: $L \to +\infty$ (i.e., $h = 1/L \to 0$), $m$ fixed.
SEM: $L$ fixed, $m \to +\infty$.

# Discrete fixed point operator

The restriction and prolongation operators are extended to $X$ as

$$R_L(u,\psi,\omega) = (\rho_L^+ u, \rho_L^- \psi, \omega), \qquad P_L(u_L,\psi_L,\omega) = (\pi_L^+ u_L, \pi_L^- \psi_L, \omega).$$

Setting $X_L := \mathbb{U}_L \times \mathbb{A}_L \times \mathbb{R}$, the discrete fixed point operator $\Phi_L = R_L \Phi P_L : X_L \to X_L$ is given by

$$\Phi_L(u_L,\psi_L,\omega) := \begin{pmatrix} \omega\rho_L^+ G(\mathcal{G}(\pi_L^+ u_L, \pi_L^- \psi_L)_\cdot \circ s_\omega) \\ \rho_L^- \mathcal{G}(\pi_L^+ u_L, \pi_L^- \psi_L)_1 \\ \omega - p(\mathcal{G}(\pi_L^+ u_L, \pi_L^- \psi_L)|_{[0,1]}) \end{pmatrix}.$$

The operator $\hat{\Phi}_L := P_L R_L \Phi : X \to X$ is, however, the one defining the discrete fixed point problem $\hat{\Phi}_L(x) = x$ needed to perform the convergence analysis.

# General discretization theory

The validity of the discretization scheme depends on the comparison between the exact and the discrete fixed point problems

$$\Phi(x) = x, \qquad \hat{\Phi}_L(x) = x.$$

The discrete problem must have a unique solution $x_L^*$. From

$$\begin{aligned} x_L^* - x^* = \;& \hat{\Phi}_L(x_L^*) - \Phi(x^*) \\ = \;& P_L R_L \Phi(x^*) - \Phi(x^*) + P_L R_L[\Phi(x_L^*) - \Phi(x^*)] \\ = \;& P_L R_L x^* - x^* \\ & + P_L R_L[\Phi(x_L^*) - \Phi(x^*) - D\Phi(x^*)(x_L^* - x^*)] \\ & + P_L R_L D\Phi(x^*)(x_L^* - x^*), \end{aligned}$$

one gets

$$\begin{aligned} [I - P_L R_L D\Phi(x^*)](x_L^* - x^*) = & \; P_L R_L x^* - x^* \\ & + P_L R_L[\Phi(x_L^*) - \Phi(x^*) - D\Phi(x^*)(x_L^* - x^*)]. \end{aligned}$$

# General discretization theory

$$[I - P_L R_L D\Phi(x^*)](x_L^* - x^*) = P_L R_L x^* - x^* \\ + P_L R_L[\Phi(x_L^*) - \Phi(x^*) - D\Phi(x^*)(x_L^* - x^*)].$$

Thus, the well-posedeness of the discrete problem and the convergence $x_L^* \to x^*$ require

- existence and uniform boundedness of $\|[I - P_L R_L D\Phi(x^*)]^{-1}\|_{X \leftarrow X}$ as $L \to +\infty$ (stability)
- $\|P_L R_L x^* - x^*\|_X \to 0$ as $L \to +\infty$ (consistency).
- $\|P_L R_L[\Phi(x_L^*) - \Phi(x^*) - D\Phi(x^*)(x_L^* - x^*)]\|_X \leq K\|x_L^* - x^*\|_X$ uniformly as $L \to +\infty$, $K$ small.

# Numerical assumptions needed by [6]

**CS1** Given $\Psi_L := I - \hat{\Phi}_L : X \to X$, $D\Psi_L$ is Lipschitz continuous at $x^*$ of some constant $\kappa_L$ in some open ball $\mathcal{B}(x^*, r_1(L))$

**CS2**

$$\lim_{L\to+\infty} \frac{\|[D\Psi_L(x^*)]^{-1}\|_{X\leftarrow X} \cdot \|\Psi_L(x^*)\|_X}{r_2(L)} = 0$$

for

$$r_2(L) = \min\left\{r_1(L), \frac{1}{2\kappa_L\|[D\Psi_L(x^*)]^{-1}\|_{X\leftarrow X}}\right\}.$$

- existence of $[D\Psi_L(x^*)]^{-1}$, as well as uniform boundedness of its norm (stability), need to be proved directly, cannot use Banach's perturbation lemma trivially
- $\lim_{L\to+\infty} \|\Psi_L(x^*)\|_X = 0$ (consistency)
- $r_1$ and $\kappa$ are indeed independent of $L$ (but $\kappa$ does depend on $m$).

[6] Maset S., *An abstract framework in the numerical solution of boundary value problems for neutral functional differential equations*, Numer. Math., 133 (2016) pp. 525–555

# Error on fixed point

**Theorem**

For $L$ large enough, $\hat{\Phi}_L$ has a (locally) unique fixed point $x_L^*$ and

$$\|x_L^* - x^*\|_X \leq 2\|[D\Psi_L(x^*)]^{-1}\|_{X\leftarrow X} \cdot \|\Psi_L(x^*)\|_X.$$

Therefore,

$$\begin{aligned}\|(v_L^*, \omega_L^*) &- (v^*, \omega^*)\|_{\mathbb{V}\times\mathbb{B}} \\ &\leq 2 \cdot \max\{\|\mathcal{G}\|_{\mathbb{V}\leftarrow\mathbb{U}\times\mathbb{A}}, 1\} \cdot \|[D\Psi_L(x^*)]^{-1}\|_{X\leftarrow X} \cdot \|\Psi_L(x^*)\|_X.\end{aligned}$$

If, in addition, $G \in C^p(\mathtt{Y}, \mathbb{R}^d)$ for some $p \geq 1$, then $u^* \in C^p([0,1], \mathbb{R}^d)$, $\psi^* \in C^{p+1}([-1,0], \mathbb{R}^d)$, $v^* \in C^{p+1}([-1,1], \mathbb{R}^d)$ and

$$\|\Psi_L(x^*)\|_X = O(h^{\min\{m,p\}}).$$

Same order as in the (experimental) literature of periodic BVPs.

# Extension to REs

$$\begin{cases} x(t) = F(\overline{x}_t), & t \in [0, \omega], \\ x(0) = x(\omega) \\ p(x|_{[0,\omega]}) = 0, \end{cases}$$

with $F(\varphi) = \int_{-\tau}^{0} K(\sigma, \varphi(\sigma))\, \mathrm{d}\sigma$.
Framework seems to work by choosing regularity $B^\infty$ for all the spaces involved.

What about $k_1 \neq 0$? References for adjoint theory of REs?

# (Present and) future work

- the theoretical convergence in case of DDEs has now been proved using the less natural BVP formulation. However, both formulations lead to fundamentally equivalent numerical methods
- extension of the convergence proof to DDEs with different type of delays (e.g., state-dependent provided that the map $v_t \mapsto \tau(v_t)$ is differentiable - work in progress)
- extension of the convergence proof to REs (work in progress), coupled systems and neutral DDEs
- application to compute periodic solutions of realistic models (e.g., Daphnia [11]), where the right-hand sides are only defined through solutions of *external* ODEs.

[11] Diekmann O., Gyllenberg M., Metz J.A.J., Nakaoka S., and de Roos A.M., *Daphnia revisited: local stability and bifurcation theory for physiologically structured population models explained by way of an example*, J. Math. Biol., 61:277-318, 2010.

Thank you for your attention

# Collocation solution

In each $[t_i, t_{i+1}]$, the collocation solution $v$ can be expressed as

$$v(t) = \sum_{j=0}^{m} v(t_{i+\frac{j}{m}}) P_{i,j}(t),$$

where

$$P_{i,j}(t) = \prod_{r=0, r\neq j}^{m} \frac{t - t_{i+\frac{r}{m}}}{t_{i+\frac{j}{m}} - t_{i+\frac{r}{m}}}, \quad j = 0, \ldots, m-1$$

and

$$t_{i+\frac{j}{m}} = t_i + \frac{j}{m}(t_{i+1} - t_i), \quad j = 0, \ldots, m-1.$$

As restated in [2], the so-called *representation points* above can be chosen independently from the collocation points.

[2] ENGELBORGHS K., LUZYANINA T., IN 'T HOUT K.J., AND ROOSE D., *Collocation methods for the computation of periodic solutions of delay differential equations*, SIAM J. Sci. Comput., 22 (2001), pp. 1593–1609.