Numerical stability analysis of linear periodic renewal equations via pseudospectral methods

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joint work with Dimitri Breda et al.

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Online Delay Days
1–2 October 2020

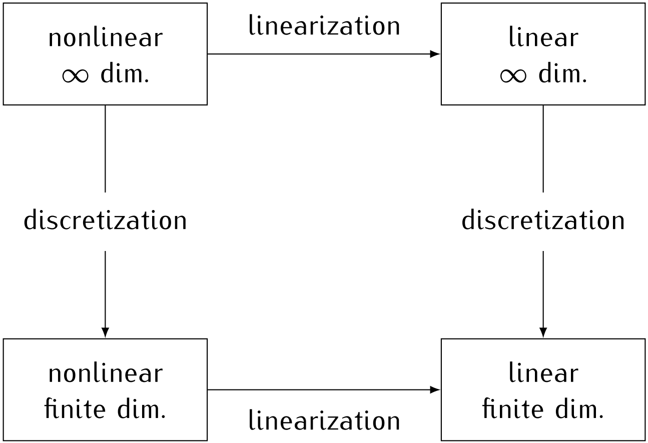
Outline

- pseudospectral method for approximating the spectra of evolution operators of linear delay equations

  joint work with Dimitri Breda
  based on work by him, Stefano Maset and Rossana Vermiglio
- extension to neutral renewal equations

  ongoing joint work with Dimitri Breda, Odo Diekmann and Sjoerd Verduyn Lunel
- linearizing nonlinear equations
  - ongoing joint work with Eugenia Franco and Francesca Scarabel
  - future work

Two complementary approaches



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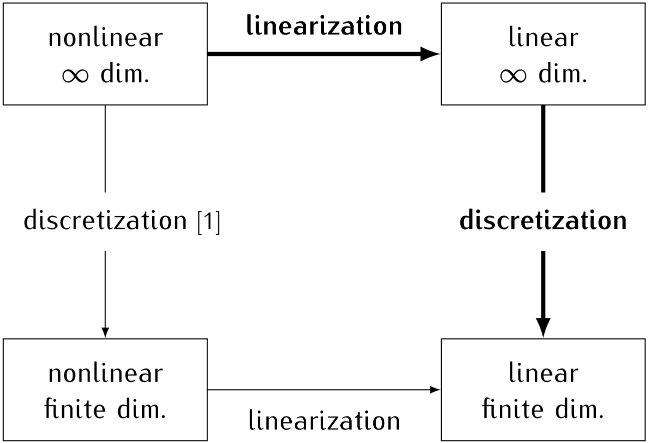


[1] Breda, Diekmann, Gyllenberg, Scarabel, and Vermiglio, *Pseudospectral discretization of nonlinear delay equations: new prospects for numerical bifurcation analysis*, SIAM J. Appl. Dyn. Sys., 15 (2016), pp. 1–23, DOI: `10.1137/15M1040931`.

Delay equations

$0 < \tau < \infty$ delay $\quad X = L^1([-\tau, 0], \mathbb{R}^{d_X})$

$d_X, d_Y \geqslant 0$ integers $\quad Y = C([-\tau, 0], \mathbb{R}^{d_Y})$

"A delay equation is a rule
for extending a function of time towards the future
on the basis of the (assumed to be) known past."

coupled RE/DDE

$$\begin{cases} x(t) = f(x_t, y_t), & f\colon X \times Y \to \mathbb{R}^{d_X} \\ y'(t) = g(x_t, y_t), & g\colon X \times Y \to \mathbb{R}^{d_Y} \end{cases}$$

Delay equations

$$0 < \tau < \infty \text{ delay} \qquad X = L^1([-\tau, 0], \mathbb{R}^{d_X})$$

$$d_X, d_Y \geqslant 0 \text{ integers} \quad Y = C([-\tau, 0], \mathbb{R}^{d_Y})$$

$$y_t(\theta) \coloneqq y(t + \theta), \quad \theta \in [-\tau, 0]$$

DDE (RFDE)

$$y'(t) = g(y_t), \quad g\colon Y \to \mathbb{R}^{d_Y}$$

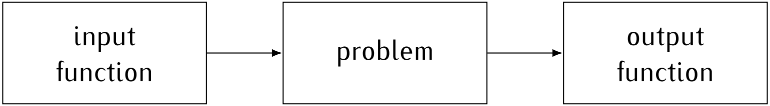
RE (VFE)

$$x(t) = f(x_t), \quad f\colon X \to \mathbb{R}^{d_X}$$

coupled RE/DDE

$$\begin{cases} x(t) = f(x_t, y_t), & f\colon X \times Y \to \mathbb{R}^{d_X} \\ y'(t) = g(x_t, y_t), & g\colon X \times Y \to \mathbb{R}^{d_Y} \end{cases}$$

Pseudospectral methods: key idea



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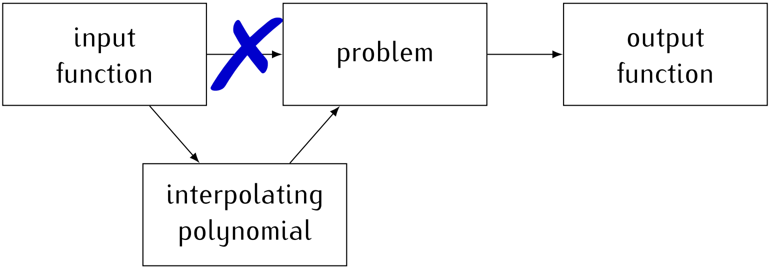
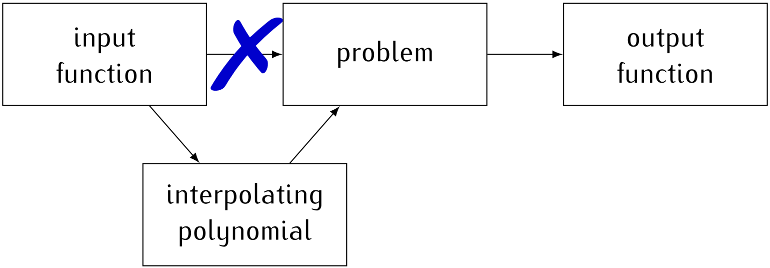


Pseudospectral methods: key idea

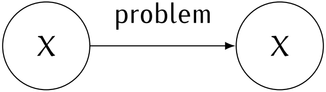


$\rightsquigarrow$ spectral accuracy [2]

with "good" nodes (e.g., Čebyšëv zeros or extrema)
smooth functions: error $= O(N^{-k})$ for every k
analytic functions: error $= O(c^N)$ for $0 < c < 1$

[2] Trefethen, *Spectral Methods in MATLAB*, Software Environ. Tools, SIAM, Philadelphia, 2000, DOI: 10.1137/1.9780898719598.

Pseudospectral methods: more details



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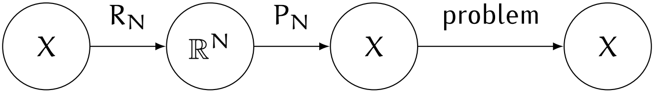
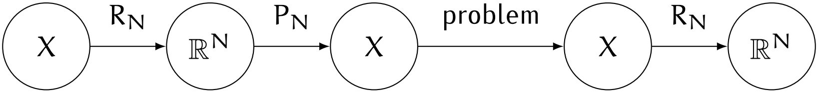


$$\varphi \longmapsto R_N\varphi \longmapsto P_N R_N \varphi \longmapsto \psi$$

Pseudospectral methods: more details



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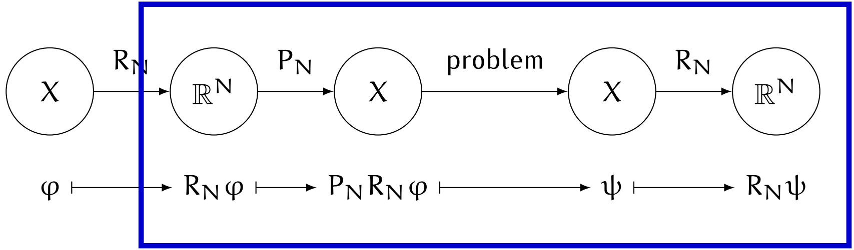

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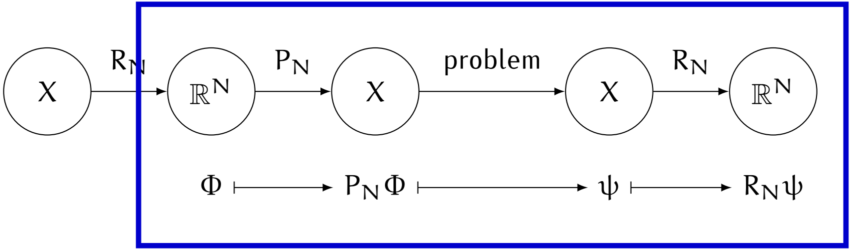

SO approach for RE: reconstructing the solution

$$\begin{cases} x(t) = \int_{-\tau}^{0} C(t,\theta) x_t(\theta)\, d\theta \\ x_0 = \varphi \end{cases}$$

$$U \coloneqq U(p, 0) \colon \varphi = x_0 \mapsto x_p$$

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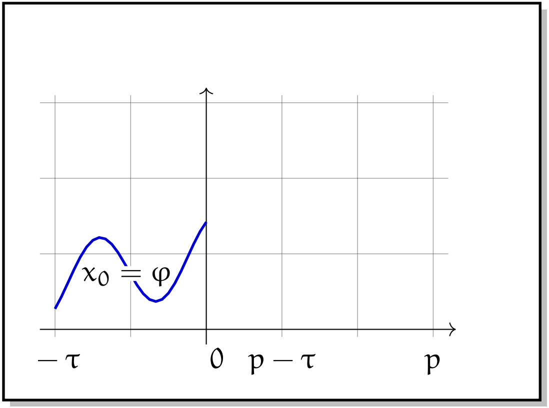

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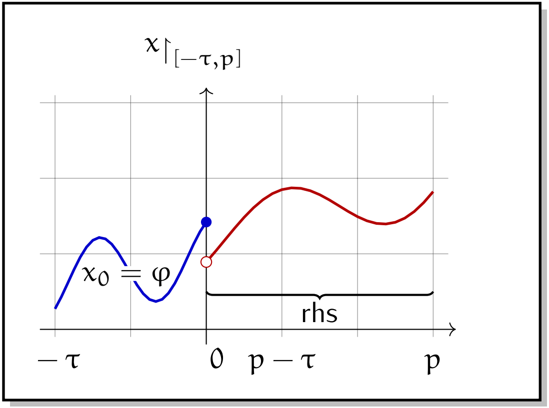

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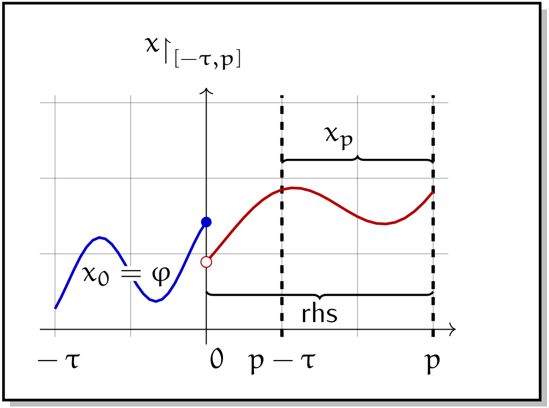

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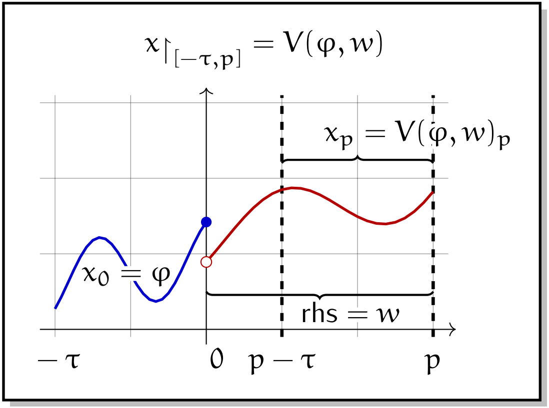

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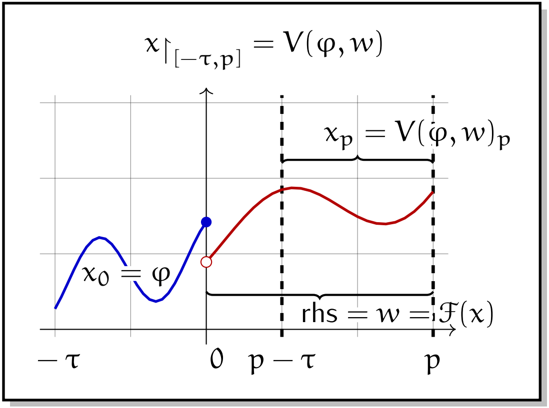
$$U := U(p,0)\colon \varphi = x_0 \mapsto x_p$$



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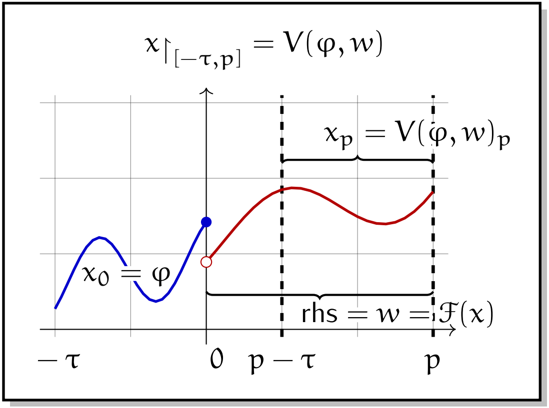

$$U\varphi = V(\varphi, w)_p$$
$$w = \mathcal{F}V(\varphi, w)$$

SO approach for DDE: reconstructing the solution

$$\begin{cases} y'(t) = L(t)y_t \\ y_0 = \psi \end{cases} \qquad U \coloneqq U(p, 0)\colon \psi = y_0 \mapsto y_p$$

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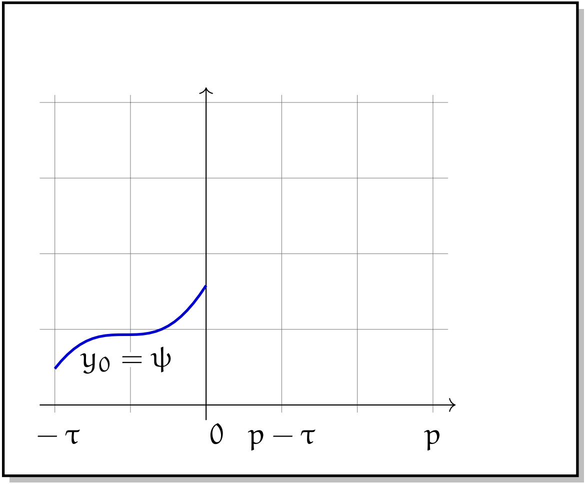

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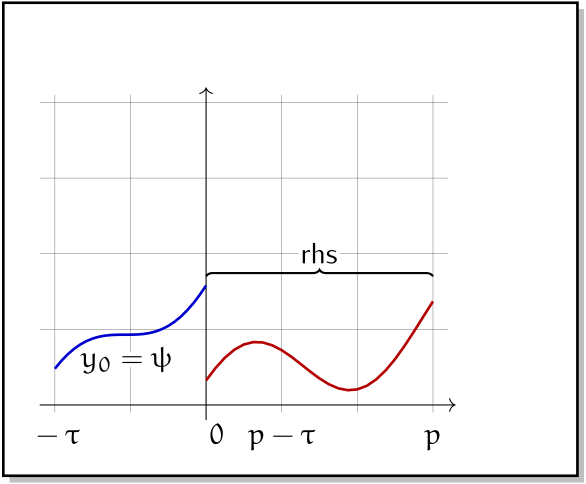

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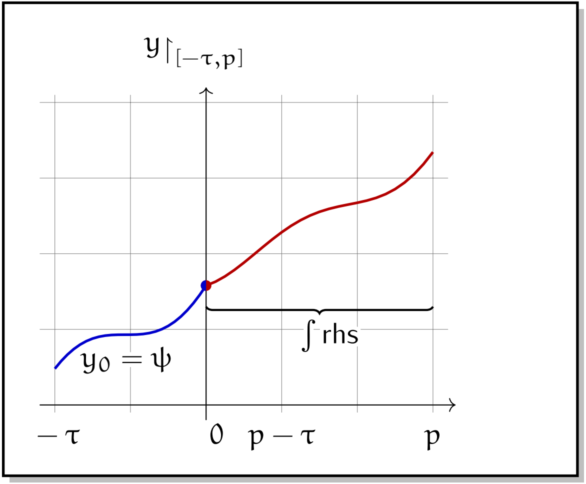

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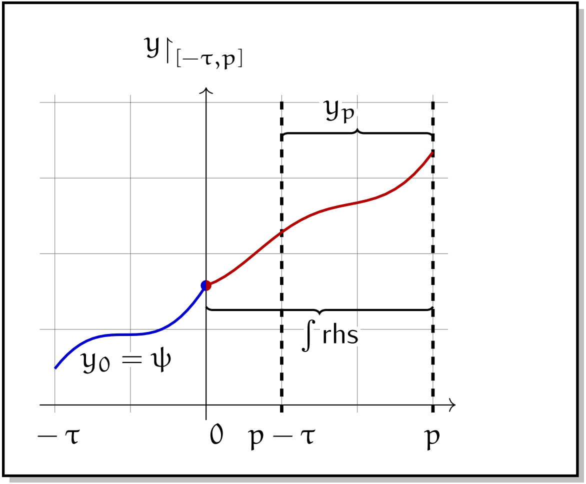

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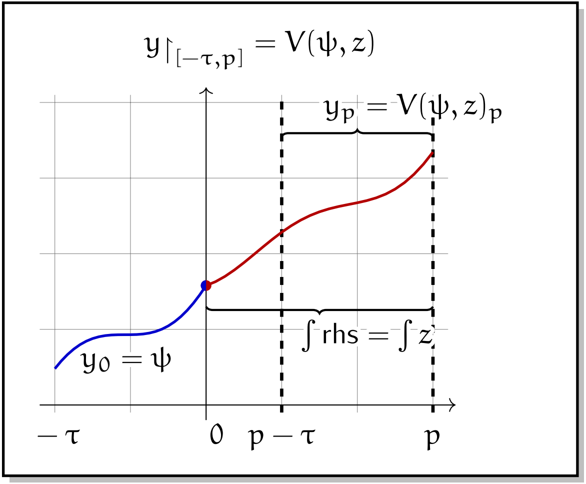

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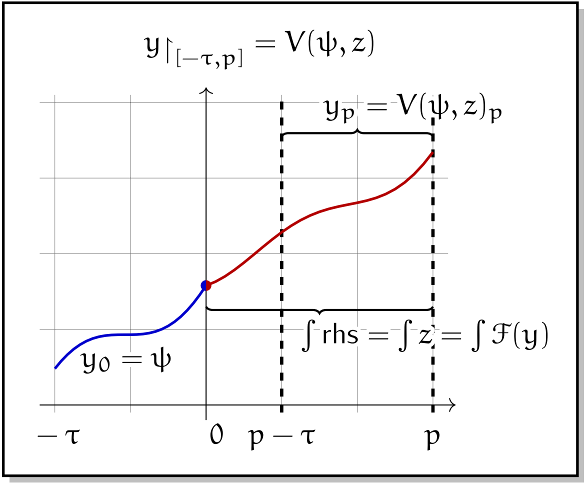

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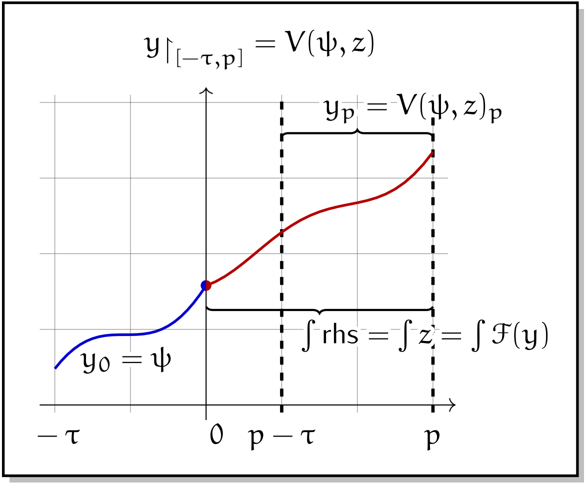

$$U\psi = V(\psi, z)_p$$
$$z = \mathcal{F}V(\psi, z)$$

SO approach: the operator V

RE

$$V(\varphi, w)(t) = \begin{cases} w(t), \ t \in]0, p] \\ \varphi(t), \ t \in [-\tau, 0] \end{cases}$$

DDE

$$V(\psi, z)(t) = \begin{cases} \psi(0) + \int_0^t z(\sigma)\, d\sigma, \ t \in [0, p] \\ \psi(t), \ t \in [-\tau, 0] \end{cases}$$

SO approach: the operator V

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coupled RE/DDE

$$V((\varphi, \psi), (w, z))(t) = \begin{cases} (w(t), \psi(0) + \int_0^t z(\sigma)\, d\sigma), \ t \in]0, p] \\ (\varphi(t), \psi(t)), \ t \in [-\tau, 0] \end{cases}$$

Pseudospectral collocation

- $M+1$ Čebyšëv extrema in $[-\tau, 0]$, R_M, P_M
- N Čebyšëv zeros in $[0, p]$, R_N^+, P_N^+
- Discretize $U \colon X \to X$

$$U\varphi = V(\varphi, w)_p$$
$$w = \mathcal{F}V(\varphi, w)$$

as $U_{M,N} \colon \mathbb{R}^{M+1} \to \mathbb{R}^{M+1}$

$$U_{M,N}\Phi = R_M V(P_M\Phi, P_N^+ W)_p$$
$$W = R_N^+ \mathcal{F}V(P_M\Phi, P_N^+ W)$$

- Compute eigenvalues of $U_{M,N}$ with standard methods.

Convergence proof

Theorem

$$\sigma(U_{M,N}) \to \sigma(U)$$

Convergence order depends on smoothness of eigenfunctions, possibly infinite (smooth functions: error $= O(M^{-k})$ for every k).

- Proved in [3] for RE, in [4] for coupled RE/DDE.
- Proof structure similar to the DDE case [5, 6], different techniques due to
  - $C \rightsquigarrow L^1$,
  - different definition of V,
  - different regularization properties of $\mathcal{F}$ and V.

[3] Breda and Liessi, *Approximation of eigenvalues of evolution operators for linear renewal equations*, SIAM J. Numer. Anal., 56 (2018), pp. 1456–1481, DOI: 10.1137/17M1140534.

[4] Breda and Liessi, *Approximation of eigenvalues of evolution operators for linear coupled renewal and retarded functional differential equations*, Ric. Mat. (2020), DOI: 10.1007/s11587-020-00513-9.

[5] Breda, Maset, and Vermiglio, *Approximation of eigenvalues of evolution operators for linear retarded functional differential equations*, SIAM J. Numer. Anal., 50 (2012), pp. 1456–1483, DOI: 10.1137/100815505.

[6] Breda, Maset, and Vermiglio, *Stability of linear delay differential equations. A numerical approach with MATLAB*, SpringerBriefs Control, Autom. and Robot., Springer, New York, 2015, DOI: 10.1007/978-1-4939-2107-2.

Convergence proof: hypotheses

- $\exists!$ of solutions (of course!).
- Čebyšëv zeros in $[0, p]$.
- Regularization properties of $\mathcal{F}V$.

Convergence proof: overview

- Well-posedness of collocation equation (N large)

$$W = R_N^+ \mathcal{F} V(\varphi, P_N^+ W)$$

- $U_{M,N} \rightsquigarrow \hat{U}_{M,N} \coloneqq P_M U_{M,N} R_M$ (same eigen-...)
- $\hat{U}_{M,N} = \mathcal{L}_M \hat{U}_N \mathcal{L}_M \rightsquigarrow \hat{U}_N$ ($M \geqslant N$ large $\Rightarrow$ same eigen-...)

- $\|\hat{U}_N - U\| \to 0$

$\Rightarrow$ $\sigma(U_{M,N}) \to \sigma(U)$, convergence order.

Tools: properties of P/R; classic results in linear algebra, functional analysis, operator theory, interpolation theory; results from [7] and [8].

[7] Krylov, *Convergence of algebraic interpolation with respect to the roots of Čebyšev's polynomial for absolutely continuous functions and functions of bounded variation*, Dokl. Akad. Nauk SSSR 107 (1956), pp. 362–365 (in Russian).

[8] Chatelin, *Spectral Approximation of Linear Operators*, Classics Appl. Math. 65, SIAM, Philadelphia, 2011, DOI: 10.1137/1.9781611970678.

Convergence proof: overview

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$$W = R_N^+ \mathcal{F} V(\varphi, P_N^+ W)$$

- $U_{M,N} \rightsquigarrow \hat{U}_{M,N} := P_M U_{M,N} R_M$ (same eigen-...)
- $\hat{U}_{M,N} = \mathcal{L}_M \hat{U}_N \mathcal{L}_M \rightsquigarrow \hat{U}_N$ ($M \geqslant N$ large $\Rightarrow$ same eigen-...)
- $U, \hat{U}_N \rightsquigarrow U\restriction_{\dots}, \hat{U}_N\restriction_{\dots}$ (same eigen-...) [DDE and coupled only]
- $\|\hat{U}_N\restriction_{\dots} - U\restriction_{\dots}\| \to 0$

$\Rightarrow$ $\sigma(U_{M,N}) \to \sigma(U)$, convergence order.

Tools: properties of P/R; classic results in linear algebra, functional analysis, operator theory, interpolation theory; results from [7] and [8].

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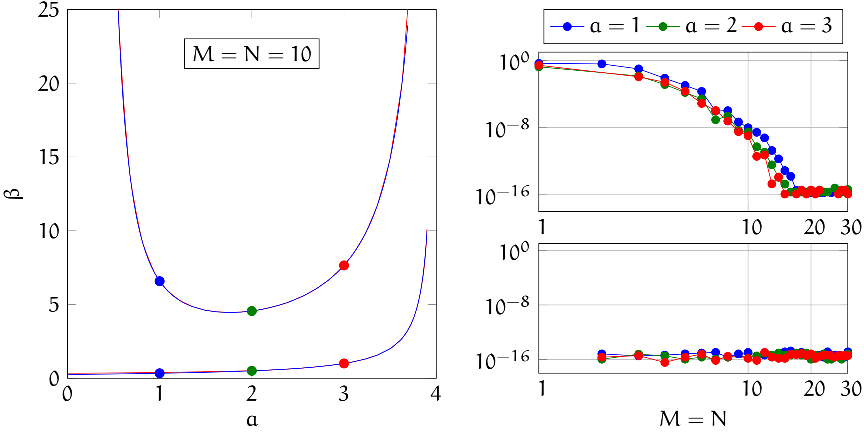
$$A(t) = \beta \int_a^4 A(t-\sigma)\, e^{-A(t-\sigma)}\, d\sigma$$



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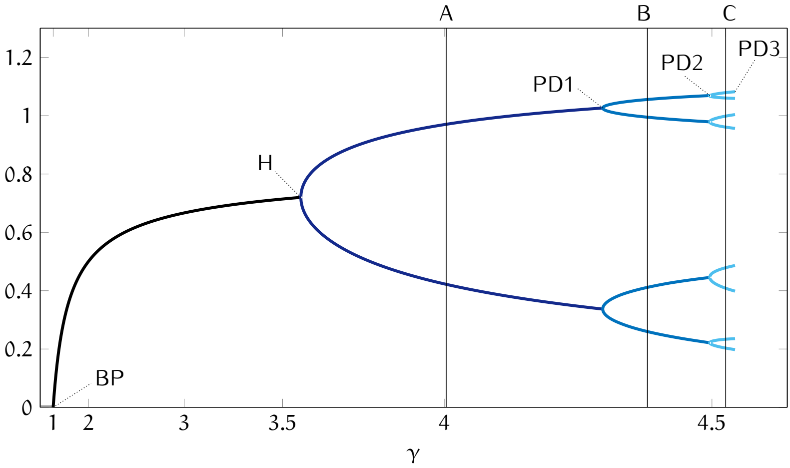
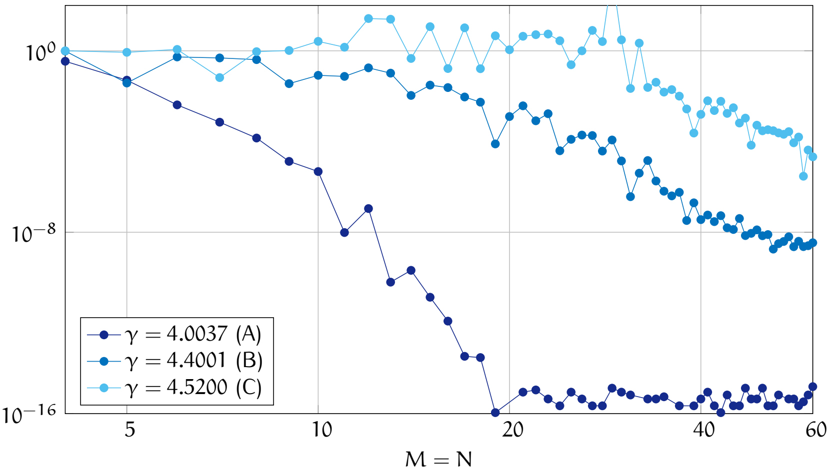
[16] Breda, Diekmann, Maset, and Vermiglio, *A numerical approach for investigating the stability of equilibria for structured population models*, J. Biol. Dyn., 7 (2013), pp. 4–20, DOI: 10.1080/17513758.2013.789562.

Special RE with quadratic nonlinearity (RE, periodic solutions) [9]

$$x(t) = \frac{\gamma}{2}\int_1^3 x(t-\sigma)\big(1 - x(t-\sigma)\big)\, d\sigma$$



[9] Breda, Diekmann, Liessi, and Scarabel, *Numerical bifurcation analysis of a class of nonlinear renewal equations*, Electron. J. Qual. Theory Differ. Equ., 65 (2016), pp. 1–24, DOI: `10.14232/ejqtde.2016.1.65`.



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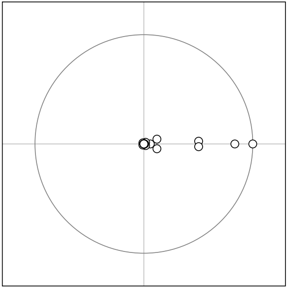
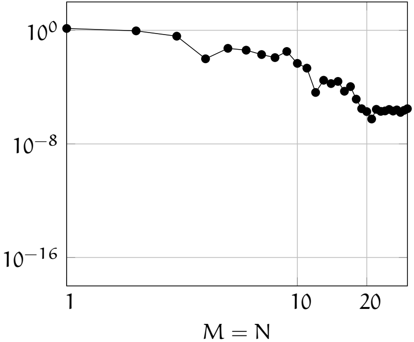
$$A(t) = \beta \int_3^4 A(t-\sigma)\, e^{-A(t-\sigma)}\, d\sigma$$

$\beta \approx 8$, numerically approximated periodic solution





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Neutral equations?

neutral DDE

vs

neutral RE

Neutral RE

$$x(t) = f(t)x(t-1)$$

f of period 1 ($k \in \mathbb{R}$, $n \in \mathbb{Z}$):

$$f \equiv k$$

$$f(t) = k + \sin(2\pi t)$$

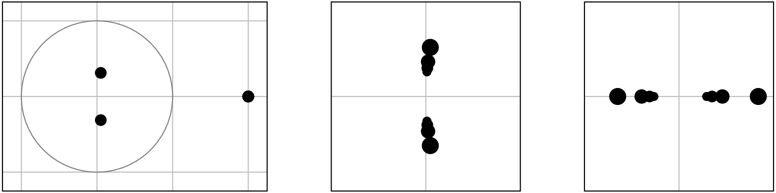
$$f(t) = k + \operatorname{sign}\left(t - \left\lfloor t + \tfrac{1}{2} \right\rfloor\right) = \begin{cases} k+1, & \text{if } t \in \left]n, n+\tfrac{1}{2}\right[\\ k, & \text{if } t = \tfrac{n}{2} \\ k-1, & \text{if } t \in \left]n+\tfrac{1}{2}, n+1\right[\end{cases}$$

$$f(t) = k + \exp(t - \lfloor t \rfloor)$$

We want to approximate $\sigma(U(1,0))$.

Neutral RE with constant coefficient

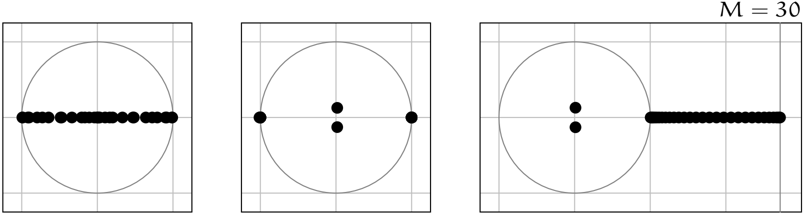
$$x(t) = 2x(t-1)$$
$$\sigma(U(1,0)) = \{2\}$$



$M = 10$ $\quad$ $M \in \{30, 60, 90, 120\}$ $\quad$ $M \in \{15, 45, 75, 105\}$

Neutral RE with nonconstant coefficient

$$x(t) = f(t)x(t-1)$$



$\sin(2\pi t)$ $\quad$ $\operatorname{sign}\left(t - \lfloor t + \frac{1}{2} \rfloor\right)$ $\quad$ $\exp(t - \lfloor t \rfloor)$

Neutral RE with nonconstant coefficient

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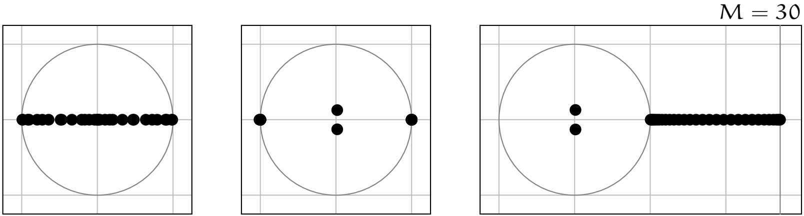


$\sin(2\pi t)$ $\quad$ $\mathrm{sign}\left(t - \lfloor t + \frac{1}{2} \rfloor\right)$ $\quad$ $\exp(t - \lfloor t \rfloor)$

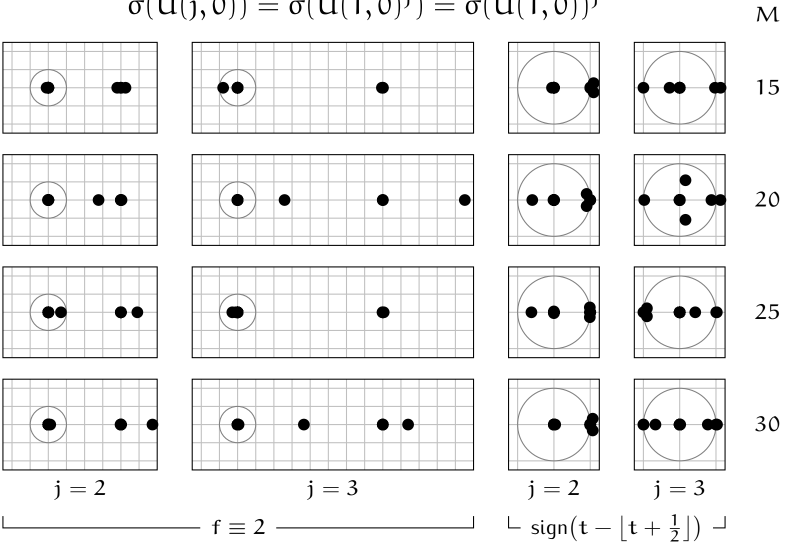
Conjecture

$$\sigma(U(\tau, 0)) = f(\mathbb{R})$$

Does $\sigma(U_{M,N}(\tau, 0))$ converge to $f(\mathbb{R})$ or $\{0\} \cup f(\mathbb{R})$?

"Polydromy" operators, or multiple periods

$$\sigma(U(j,0)) = \sigma(U(1,0)^j) = \sigma(U(1,0))^j$$



Approximating equations

$$x(t) = \frac{f(t)}{2\epsilon} \int_{-\tau-\epsilon}^{-\tau+\epsilon} x(t+\theta)\, d\theta$$

$$x'(t) = \frac{f(t)}{2\epsilon} (x(t-\tau+\epsilon) - x(t-\tau-\epsilon))$$

A key role for 0?

	multiplicity of 0 in $\sigma(\cdots)$		
j	$U_{M,N}(j,0)$	$U_{R,\epsilon,M,N}(j,0)$	$U_{D,\epsilon,M,N}(j,0)$
1	2	$M+2$	M
2	$0.70M$	$0.70M$	$0.66M$
3	$0.81M$	$0.81M$	$0.77M$
4	$0.86M$	$0.86M$	$0.82M$
5	$0.89M$	$0.89M$	$0.85M$

$$U_{M,N}(j,0) \in \mathbb{R}^{(M+1)\times(M+1)}$$
$$U_{R/D,\epsilon,M,N}(1,0) \in \mathbb{R}^{(2M+1)\times(2M+1)}$$
$$U_{R/D,\epsilon,M,N}(j,0) \in \mathbb{R}^{(M+1)\times(M+1)} \quad \text{if } j>1$$

Convergence of the operators

$$
\begin{array}{ccccc}
U_{R,\epsilon,M,N}(p,0) & \xrightarrow[\epsilon \to 0]{} & U_{M,N}(p,0) & \xleftarrow[\epsilon \to 0]{} & U_{D,\epsilon,M,N}(p,0) \\
\downarrow {\scriptstyle M \to +\infty} & & \downarrow {\scriptstyle M \to +\infty} & & \downarrow {\scriptstyle M \to +\infty} \\
U_{R,\epsilon}(p,0) & \xrightarrow[\epsilon \to 0]{} & U(p,0) & \xleftarrow[\epsilon \to 0]{} & U_{D,\epsilon}(p,0)
\end{array}
$$

Linearizing RE

$$x(t) = F(x_t)$$

Examples of F:

$$(1)\quad L^1 \xrightarrow[\text{bounded}]{\text{linear/affine}} \mathbb{R}^d \xrightarrow[\text{differentiable}]{\text{nonlinear}} \mathbb{R}^d \quad \checkmark$$

$$(2)\quad L^1 \xrightarrow[\text{with } f \text{ differentiable}]{N_f \text{ Nemyckij}} L^1 \xrightarrow[\text{bounded}]{\text{linear/affine}} \mathbb{R}^d \quad \times$$

$$(3)\quad L^1 \xrightarrow[\text{bounded}]{\text{linear/affine}} C \xrightarrow[\text{differentiable}]{\text{nonlinear}} \mathbb{R}^d \quad \checkmark$$

Trick in [10] to fix (2) for equilibria.

Periodic solutions?
Coupled RE/DDE?

[10] Diekmann, Getto, and Gyllenberg, *Stability and bifurcation analysis of Volterra functional equations in the light of suns and stars*, SIAM J. Math. Anal., 39 (2008), pp. 1023–1069, DOI: 10.1137/060659211.

Linearizing with automatic differentiation

Computing Fréchet derivatives: inconvenient, difficult, error-prone.

Automatic differentiation (with no truncation error!):

- developed for derivatives of real functions [11],
- extended to Fréchet derivatives of operators on C [12].

(Almost) ready to use for DDE?

For RE we need to

- adapt the differentiation rules,
- recognize (2) and automatically apply the trick.

[11] Griewank and Walther, *Evaluating Derivatives: Principles and Techniques of Algorithmic Differentiation*, 2nd ed., SIAM, Philadelphia, 2008, DOI: `10.1137/1.9780898717761`.

[12] Birkisson and Driscoll, *Automatic Fréchet differentiation for the numerical solution of boundary-value problems*, ACM Trans. Math. Software, 38 (2012), pp. 26:1–26:29, DOI: `10.1145/2331130.2331134`.

Future perspectives

- Neutral DDE (reformulation of $\mathcal{U}$ is difficult!).
- Apply to realistic models, e.g., *Daphnia* [13].
- Extend ideas to Ljapunov exponents (see, e.g., [15]).

Daphnia magna, female [14]

[13] Diekmann, Gyllenberg, Metz, Nakaoka, and De Roos, Daphnia *revisited: local stability and bifurcation theory for physiologically structured population models explained by way of an example*, J. Math. Biol., 61 (2010), pp. 277–318, DOI: `10.1007/s00285-009-0299-y`.

[14] Watanabe, *Female adult of the water flea* Daphnia magna, PLoS Genetics, 7 (2011), issue image, DOI: `10.1371/image.pgen.v07.i03`.

[15] Breda and Van Vleck, *Approximating Lyapunov exponents and Sacker–Sell spectrum for retarded functional differential equations*, Numer. Math., 126 (2014), pp. 225–257, DOI: `10.1007/s00211-013-0565-1`.

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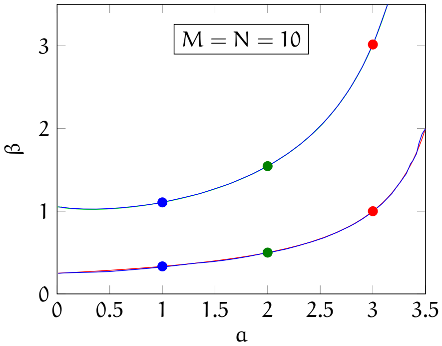
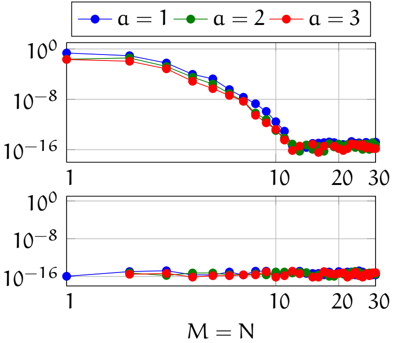
Thank you!

Simplified *Daphnia* (coupled RE/DDE, equilibria) [16, 4, PhD thesis]

$$\begin{cases} b(t) = \beta S(t) \displaystyle\int_a^4 b(t-\sigma)\,d\sigma \\ S'(t) = S(t)\big(1 - S(t)\big) - S(t) \displaystyle\int_a^4 b(t-\sigma)\,d\sigma \end{cases}$$



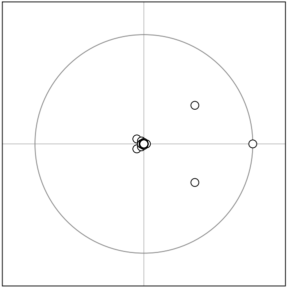


[4] Breda and Liessi, *Approximation of eigenvalues of evolution operators for linear coupled renewal and retarded functional differential equations*, Ric. Mat. (2020), DOI: 10.1007/s11587-020-00513-9.

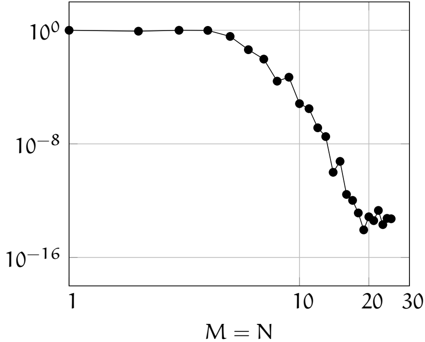
[16] Breda, Diekmann, Maset, and Vermiglio, *A numerical approach for investigating the stability of equilibria for structured population models*, J. Biol. Dyn., 7 (2013), pp. 4–20, DOI: 10.1080/17513758.2013.789562.

$$\begin{cases} x(t) = -\int_{-\pi}^{0} (x^2(t+\theta) + y^2(t+\theta)) y(t+\theta)\, d\theta + y\left(t - \frac{\pi}{2}\right) \\ y'(t) = \int_{-\frac{\pi}{2}}^{0} (x^2(t+\theta) + y^2(t+\theta)) x(t+\theta)\, d\theta + y(t-\pi) \end{cases}$$

$$(\bar{x}(t), \bar{y}(t)) = (\cos(t), \sin(t))$$



and 508.94

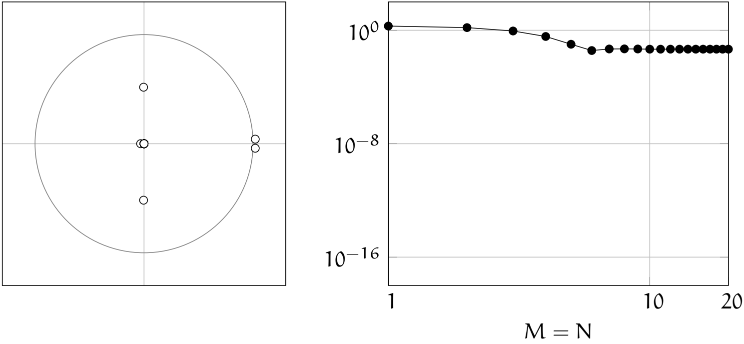


[4] Breda and Liessi, *Approximation of eigenvalues of evolution operators for linear coupled renewal and retarded functional differential equations*, Ric. Mat. (2020), DOI: 10.1007/s11587-020-00513-9.

Simplified *Daphnia* (coupled RE/DDE, periodic solutions) [16, PhD thesis]

$$\begin{cases} b(t) = \beta S(t) \displaystyle\int_a^4 b(t-\sigma)\,d\sigma \\ S'(t) = S(t)\big(1 - S(t)\big) - S(t) \displaystyle\int_a^4 b(t-\sigma)\,d\sigma \end{cases}$$

$a = 2$, $\beta \approx 3$, numerically approximated periodic solution



[16] Breda, Diekmann, Maset, and Vermiglio, *A numerical approach for investigating the stability of equilibria for structured population models*, J. Biol. Dyn., 7 (2013), pp. 4–20, DOI: 10.1080/17513758.2013.789562.