Polyfold methods for Delay equations

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joint work with Peter Albers

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- I am a PhD student at Heidelberg University, working in the Symplectic Geometry group. All that I will present today is joint work with Peter Albers.
- Today want to discuss a result about delay equations which we found using methods from symplectic topology.
- We do not have a strong background on DDEs, but view delay equations and dynamical systems from a geometric viewpoint.
- I hope we can find a common language to discuss the connections between polyfolds and DDEs!
- If you have any questions during the talk, do not hesitate to ask!

Plan for this talk:

1. Idea of Polyfold theory
2. Setting and main result
3. The geometric approach
4. Classical differentiability
5. Definitions and sc-smoothness
6. Applying the M-polyfold IFT

Idea of Polyfold theory

Polyfold theory

- was developed by H. Hofer, K. Wysocki and E. Zehnder
- is used for the study of moduli spaces of J-holomorphic curves (maps $u : (\Sigma, i) \to (M, J)$ that satisfy the PDE $u \circ i = J \circ u$, modulo reparametrization)
- was made to overcome the following problems that frequently occur in the study of moduli spaces:
  - varying domains
  - varying automorphism groups
  - transversality issues
  - **non-smoothness of reparametrization maps**
- is described in full detail in the book “Polyfold and Fredholm theory” by HWZ which is available on Arxiv.

Main ingredients of polyfold theory:

- **sc-Banach spaces**
- **sc-differentiability**: a new (weaker) sense of differentiability for maps between sc-Banach spaces

} **sc-calculus**

- **M-polyfolds**: topological spaces that locally look like open subsets of (sc-retracts in) these sc-Banach spaces, with sc-smooth transition maps
- **polyfolds**: M-polyfolds with some additional orbifold behaviour

Important results:

- regularization scheme: "every intersection can be made transverse by suitable small perturbation"
- an implicit function theorem (IFT) for sc-smooth sc-Fredholm sections

Setting and main theorem

$X : S^1 \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ smooth, time-dependent vector field on $\mathbb{R}^n$ (1-periodic in time)

$\tau \in \mathbb{R}$ parameter, called "delay"

We study 1-periodic solutions $x : S^1 \longrightarrow \mathbb{R}^n$ of the following equation:

$$\dot{x}(t) = X_t\big(x(t-\tau)\big)$$

Note: We fix the period and want to vary the delay.

Idea: If there is a "nice" solution without delay, then there should be a whole family of solutions for small delays.

But what do I mean by "nice"?

Definition

Let $\Phi_X^t : \mathbb{R}^n \to \mathbb{R}^n$ denote the flow of X. A 1-periodic solution x of $\dot{x}(t) = X_t\big(x(t)\big)$ is **non-degenerate** if the linear map $\mathrm{d}\Phi_X^1\big(x(0)\big) : \mathbb{R}^n \to \mathbb{R}^n$ does not have 1 as an eigenvalue.

Theorem (Albers–S.)

Assume that x_0 is a non-degenerate 1-periodic orbit of the vector field X.

Then for every small enough $\tau \in \mathbb{R}$ there exists a (locally unique) solution $x_\tau \in \mathcal{C}^\infty(S^1, \mathbb{R}^n)$ of the delay equation

$$\dot{x}(t) = X_t\big(x(t-\tau)\big).$$

The parametrization $\tau \longmapsto x_\tau$ is smooth.

The result can be generalized to finitely many discrete delays, and also to suitable delay equations on orientable manifolds.

In this talk: explain how to prove the theorem via the M-polyfold IFT

The geometric approach

Solutions are zeros of the map

$$\begin{aligned} s : \mathbb{R} \times \mathcal{C}^\infty(S^1, \mathbb{R}^n) &\longrightarrow \mathcal{C}^\infty(S^1, \mathbb{R}^n) \\ (\tau, x) &\longmapsto \dot{x} - X\big(x(\cdot - \tau)\big). \end{aligned}$$

We start with $s(0, x_0) = 0$ and claim that the zero set near $(0, x_0)$ carries the structure of a 1-dimensional smooth manifold (and is not contained in $\{0\} \times \mathcal{C}^\infty(S^1, \mathbb{R}^n)$).

$\Longrightarrow$ want to use an implicit function theorem (IFT)
i.e. something like "the zero set of a smooth transverse Fredholm section in a Banach bundle is a smooth manifold"

Problem: The shift map $(\tau, x) \longmapsto x(\cdot - \tau)$ is not smooth.

$\Longrightarrow$ use sc-calculus and the M-polyfold IFT instead

Classical differentiability

From now on denote $H_m := W^{m,2}(S^1, \mathbb{R}^n)$.

For $\tau \in \mathbb{R}$ and $x \in H_0$ define

$$\varphi(\tau, x) := x(\cdot - \tau) \in H_0.$$

Note: If $x \in H_m$, then $\varphi(\tau, x) \in H_m$ with $\|\varphi(\tau, x)\|_{H_m} = \|x\|_{H_m}$.

Fact

- *The shift map*

$$\begin{aligned} \varphi : \mathbb{R} &\longrightarrow \mathcal{L}(H_0, H_0) \\ \tau &\longmapsto \big(x \mapsto x(\cdot - \tau)\big) \end{aligned}$$

  is not continuous when the target space carries the operator topology.

- *The shift map*

$$\varphi : \mathbb{R} \times H_1 \longrightarrow H_0$$

  is continuously differentiable with

$$\mathrm{d}\varphi(\tau, x)(T, \hat{x}) = \varphi(\tau, \hat{x}) - T \cdot \varphi(\tau, \dot{x}).$$

We can write the map that cuts out our solution space as

$$\begin{aligned} s : \mathbb{R} \times H_1 &\longrightarrow H_0 \\ (\tau, x) &\longmapsto \dot{x} - X\big(\varphi(\tau, x)\big). \end{aligned}$$

Thus:

- s is continuously differentiable.
- It is not smooth on any fixed $\mathbb{R} \times H_m$.

This means:

- We could use a $\mathcal{C}^1$-version of the IFT for Banach spaces and get a $\mathcal{C}^1$-manifold of smooth solutions $(x_\tau)_\tau$.
- This can only provide a $\mathcal{C}^1$-parametrization $\tau \longmapsto x_\tau$, not smooth.
- If as domain we take $\mathbb{R} \times H_2$, we gain another derivative of s, but we loose the Fredholm property.

But: The setting of sc-calculus comes naturally and solves the problem!

Definitions and sc-smoothness

Definition (Hofer–Wysocki–Zehnder)

A **sc-Banach space E** is a Banach space E_0 with a filtration by subspaces

$$E_0 \supseteq E_1 \supseteq E_2 \supseteq \dots$$

such that

- each E_m is a Banach space on its own
- each embedding $E_{m+1} \hookrightarrow E_m$ is compact and dense.

Remark

- If E_0 is finite-dimensional, then $E_0 = E_1 = E_2 = \dots$.
- If E_0 is infinite-dimensional, then $E_0 \neq E_1 \neq E_2 \neq \dots$.

By E^1 denote the space E_1 with the induced sc-structure:

$$E^1_m := E_{m+1}$$

In our setting:

$$\mathsf{H} = \left(H_m = W^{m,2}(S^1, \mathbb{R}^n)\right)_{m\geq 0}$$
$$\mathsf{H}^1 = \left(H^1_m = W^{m+1,2}(S^1, \mathbb{R}^n)\right)_{m\geq 0}$$

$\mathbb{R}$ with constant sc-structure

Definition (HWZ)

A map $f : \mathrm{E} \longrightarrow \mathrm{F}$ between sc-Banach spaces is sc^0 or **sc-continuous** if

- for all $m \geq 0$ it is $f(E_m) \subseteq F_m$, and
- the induced map $f|_{E_m} : E_m \longrightarrow F_m$ is continuous.

Definition (HWZ)

A sc-continuous map $f : \mathsf{E} \longrightarrow \mathsf{F}$ is sc^1 or **sc-differentiable** if

- for every $x \in E_1$ there is a bounded linear operator $\mathrm{d}f(x) : E_0 \longrightarrow F_0$ such that

$$\lim_{\|h\|_{E_1} \to 0} \frac{1}{\|h\|_{E_1}} \cdot \|f(x+h) - f(x) - \mathrm{d}f(x)h\|_{F_0} = 0$$

- the tangent map

$$\begin{aligned} Tf : \mathsf{E}^1 \oplus \mathsf{E} &\longrightarrow \mathsf{F}^1 \oplus \mathsf{F} \\ (x, h) &\longmapsto \big(f(x), \mathrm{d}f(x)h\big) \end{aligned}$$

  is sc-continuous.

Remark

- derivative exists only at the 1-level, but it is a bounded linear operator between 0-levels
- derivative may not be continuous in operator norms, only after evaluation on every level
- in finite dimensions get back the usual notions of differentiability and smoothness

Definition (HWZ)

Inductively, f is sc^k or **k times sc-differentiable** if it is sc-differentiable and its tangent map is (k-1) times sc-differentiable.

It is sc$^\infty$ or **sc-smooth** if it is sc^k for all $k \in \mathbb{N}$.

Theorem (HWZ)

The chain rule holds in sc-calculus.

Theorem (Frauenfelder–Weber)

The shift map

$$\begin{aligned} \varphi : \mathbb{R} \times \mathsf{H} &\longrightarrow \mathsf{H} \\ (\tau, x) &\longmapsto x(\cdot - \tau) \end{aligned}$$

is sc-smooth.

Corollary (Albers–S.)

The map

$$\begin{aligned} s : \mathbb{R} \times \mathsf{H}^1 &\longrightarrow \mathsf{H} \\ (\tau, x) &\longmapsto \dot{x} - X\big(\varphi(\tau, x)\big) \end{aligned}$$

is sc-smooth.

Applying the M-polyfold IFT

Recall:

Theorem (IFT for Banach bundles)

Assume that

- *we have a Banach space bundle over a Banach manifold,*
- f *is a smooth section,*
- f *has the Fredholm property and*
- f *is transverse to the zero section.*

Then the zero set of f *is a smooth manifold of dimension equal to the Fredholm index of* f*.*

Theorem (IFT for M-polyfold bundles)

Assume that

- *we have a tame strong M-polyfold bundle admitting sc-smooth bump functions,*
  our setting: the trivial sc-Hilbert space bundle
  $$\mathbb{R} \times \mathrm{H}^1 \times \mathrm{H} \longrightarrow \mathbb{R} \times \mathrm{H}^1$$
- *f is a sc-smooth section,*
  our setting: $s : \mathbb{R} \times \mathrm{H}^1 \longrightarrow \mathrm{H}$
- *f has the sc-Fredholm property and*
  our case: some work!
- *f is in good position*
  our case: need transversality

Then the zero set of f is a smooth manifold of dimension equal to the Fredholm index of f.

Indeed, we could prove the following:

Theorem (Albers–S.)

1. *$s : \mathbb{R} \times \mathrm{H}^1 \longrightarrow \mathrm{H}$ is a sc-Fredholm section of index 1.*
2. *If x_0 is a non-degenerate periodic orbit of X, then $\mathrm{d}s(0, x_0) : \mathbb{R} \times \mathrm{H}^1 \longrightarrow \mathrm{H}$ is surjective.*

This together with the M-polyfold IFT implies our main theorem:

Theorem (Albers–S.)

Assume that x_0 is a non-degenerate 1-periodic orbit of the vector field X.

Then for every small enough $\tau \in \mathbb{R}$ there exists a (locally unique) solution $x_\tau \in \mathcal{C}^\infty(S^1, \mathbb{R}^n)$ of the delay equation

$$\dot{x}(t) = X_t\big(x(t-\tau)\big).$$

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